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A general formula of the effective potential in 5D $SU(N)$ gauge theory on orbifold

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Abstract: We show a general formula of the one loop effective potential of the 5D $SU(N)$ gauge theory compactified on an orbifold, S^1/Z_2. The formula shows the case when there are fundamental, (anti-)symmetric tensor and adjoint representational bulk fields. Our calculation method is also applicable when there are bulk fields belonging to higher dimensional representations. The supersymmetric version of the effective potential with Scherk-Schwarz breaking can be obtained straightforwardly. We also show some examples of effective potentials in $SU(3)$, $SU(5)$ and $SU(6)$ models with various boundary conditions, which are reproduced by our general formula.

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1. Introduction

More and more people pay attention to gauge theories in higher dimensions. Especially the orbifold compactification of the extra dimensional spaces have been studied by many people[1, 2]. When the gauge fields spread in the higher dimensions, their extra dimensional components are regarded as scalar fields below the compactification scale. The zero mode of these scalar fields are physical degrees of freedom (d.o.f.), which are so-called Wilson line phases. One of the most interesting examples of using Wilson line d.o.f. for the model building is to regard them as Higgs fields in the 4D effective theory. This idea is so-called gauge-Higgs unification[3, 4, 5, 6, 7, 8, 9, 10, 11]. Here the "adjoint Higgs fields" can be induced through the S^1 compactification in 5D theory, while the "Higgs doublet fields" can be induced through the orbifold compactifications. Another example is considered in Refs.[12], where the gauge coupling unification is realized due to the effects of Wilson line d.o.f. in the non-supersymmetric (SUSY) grand unified theory (GUT).

We should notice that it is very important to calculate the one loop effective potential of the Wilson line d.o.f. in order to determine the vacuum. The tree level potential can not determine the vacuum due to the existence of flat directions of the Wilson line phases. The true vacuum can be determined through the analysis of the effective potential including the quantum corrections. For example, we can know whether the color is conserved or not through the analysis of the one loop effective potential. (For the analyses in 5D $SU(5)$ GUT on S^1/Z_2, see Ref.[2].) We can also estimate the finite masses of the Wilson line d.o.f. through the 4D effective potential which include radiative corrections. In general the

Wilson line d.o.f. receive the quantum corrections, and obtain the finite masses of the order of the compactification scale. However, the quantitative estimation of the finite masses is not possible until we calculate the effective potential including the quantum corrections. Thus, in order to determine the vacuum and estimate finite masses of Wilson line d.o.f., the calculation of the one loop effective potential is strongly needed. (In $SU(3)$ and $SU(6)$ gauge-Higgs unification models, this kind of analysis had been done, and we found that the suitable electro-weak symmetry breaking can be realized dynamically[13].)

In this paper we show a general formula of the one loop effective potential of the 5D $SU(N)$ gauge theory compactified on an orbifold, S^1/Z_2. Although the formula only shows the case when there are fundamental, (anti-)symmetric tensor and adjoint representational bulk fields, our calculation method is also applicable when there are bulk fields belonging to higher dimensional representations. The SUSY version of the effective potential with Scherk-Schwarz (SS) breaking[14, 15, 16] can be obtained straightforwardly. We also show some examples of effective potentials in cases of one VEV with $P = P'$ (section 3), two VEVs with $P = P'$ (section 4.1), three VEVs with $P = P'$ (section 4.2) and one VEV with $P \neq P'$ (section 4.3), all of which are reproduced by our general formula.

2. The calculation method

In the D dimensional gauge theories, the gauge field, A_M, has the indices of the 4D spacetime, $M = 0$ - 3, and extra dimensional directions, $M = 5, 6, \cdots, D$. The components of the gauge field of extra dimensional coordinates appear as scalar fields in the 4D effective theory below the compactification scale. The zero mode components of these scalar fields, called Wilson line phases, are physical d.o.f., and the estimation of the quantum corrections of them is needed in order to determine the vacuum and finite masses. Here let us consider the 5D gauge theory for simplicity, and show the simple calculation method of evaluating the one loop effective potential of the zero mode of the 5th component of the 5D gauge field, A_5. Since our calculation method only depends on the group theoretical analysis, it is available for more than 5 dimensional gauge theories.

The effective potential at one loop level in a constant background gauge field, A_5, can be obtained by calculating the eigenvalues of $D_M(A_5)^2 = \partial_\mu^2 - D_y(A_5)^2$, where $D_M(A_5)$ is the covariant derivative and y denotes the 5th coordinate. The effective potential induced from the gauge and ghost, fermion and scalar are given by

$$V_{\text{eff}}[A_5]^{g+gh} = -(D-2)\frac{i}{2}\text{Trln}D_M D^M, \tag{2.1}$$

$$V_{\text{eff}}[A_5]^{\text{fermion}} = f(D)\frac{i}{2}\text{Trln}D_M D^M, \tag{2.2}$$

$$V_{\text{eff}}[A_5]^{\text{scalar}} = -2\frac{i}{2}\text{Trln}D_M D^M, \tag{2.3}$$

respectively, where $f(D) = 2^{[D/2]}$. The difference of the gauge- and ghost-, fermion- and scalar-contributions to the effective potential are only coming from representations and coefficients (numbers of d.o.f.) in Eqs.(2.1)-(2.3). For an adjoint representational field, the

eigenvalues are obtained by diagonalizing the bilinear form, $\mathrm{tr}\,(BD_y(A_5)D_y(A_5)B)$, where B is an adjoint representational field, as

$$-\mathrm{tr}\,(BD_y(A_5)D_y(A_5)B) \sim \mathrm{tr}(\partial_y B + ig[A_5, B])^2. \tag{2.4}$$

Here, $[A_5, B]$ is also written as $ad(A_5)B$. Once the vacuum expectation value (VEV) of A_5 is determined as $\langle A_5 \rangle = aT^a$, the $U(1)$ direction in the group space is fixed. The $ad(T^a)$ is the charge operator of this $U(1)$ that is generated by T^a. Thus, all we have to know is the charges of this $U(1)$ in order to calculate the eigenvalues in Eq.(2.4). Also, for other representations, the eigenvalues of $D_M(A_5)^2 = \partial_\mu^2 - D_y(A_5)^2$ can be calculated by their $U(1)$ charges. We can calculate the effective potential in the simple way without carrying out complicated calculations of the commutation relations by use of the structure constants. In the following sections, we consider the 5D $SU(N)$ gauge theory compactified on S^1/Z_2*. In sections 3 and 4, we show the effective potential in some models by use of our simple calculation method. Then in section 5, we show the general formula of the effective potential of 5D $SU(N)$ gauge theory compactified on S^1/Z_2.

3. A simple example

At first, we illustrate this calculation method by use of a concrete simple example. Consider a $SU(3)$ model compactified on S^1/Z_2. We adopt boundary condition as

$$P = P' = \mathrm{diag}(+, -, -), \tag{3.1}$$

which is analyzed in Ref.[7, 8, 17]. P (P') is the operator of Z_2 transformation, $y \to -y$ $(\pi R + y \to \pi R - y)$. R is the radius of the compactification scale. Under these parities, A_μ and A_5 transform as

$$(P, P')(A_\mu) = \left(\begin{array}{c|cc} (+,+) & (-,-) & (-,-) \\ \hline (-,-) & (+,+) & (+,+) \\ (-,-) & (+,+) & (+,+) \end{array}\right), \tag{3.2}$$

$$(P, P')(A_5) = \left(\begin{array}{c|cc} (-,-) & (+,+) & (+,+) \\ \hline (+,+) & (-,-) & (-,-) \\ (+,+) & (-,-) & (-,-) \end{array}\right), \tag{3.3}$$

which suggest $SU(3)$ is broken to $SU(2) \times U(1)$. The Dirac fermion, ψ, and complex scalar, ϕ, transform as

$$\phi(x, -y) = \eta T[P]\phi(x, y) \quad , \quad \phi(x, \pi R - y) = \eta' T[P']\phi(x, \pi R + y), \tag{3.4}$$

$$\psi(x, -y) = \eta T[P]\gamma^5\psi(x, y) \quad , \quad \psi(x, \pi R - y) = \eta' T[P']\gamma^5\psi(x, \pi R + y) \quad , \tag{3.5}$$

under the parity operators. $T[P]$ denotes an appropriate representation matrix, for example, when ψ belongs to the fundamental or adjoint representation, $T[P]\psi$ means $P\psi$ or

*This calculation method is also available in the case of 5D coordinate being compactified on S^1[5]. However, the advantage of our calculation method in S^1 case is not large as that in S^1/Z_2 case. Thus, we consider only S^1/Z_2 case in this paper.

$P\psi P^\dagger$, respectively. The parameters, η and η', are like *intrinsic* Parity eigenvalues, which take ± 1.

Equation (3.3) suggests that there is the Wilson line d.o.f. as a doublet of remaining $SU(2)$ gauge symmetry†. We can denote the VEV of them as

$$\langle A_5 \rangle = \frac{1}{gR} \sum_a a_a \frac{\lambda_a}{2} \to \frac{1}{2gR} \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ a & 0 & 0 \end{pmatrix}, \tag{3.6}$$

by utilizing the residual global symmetry. Here, g is the 5D gauge coupling constant. The important point is that the above VEV is proportional to one generator of $SU(2)_{13}$ that operates on the 2×2 submatrix of $(1,1)$, $(1,3)$, $(3,1)$ and $(3,3)$ components. Hereafter, we take the notation that $SU(2)_{ij}$ operates on the 2×2 submatrix of (i,i), (i,j), (j,i) and (j,j) components.

Now let us calculate the effective potential of this $SU(3)$ model induced from, for examples, an adjoint and a fundamental representational fields by using this calculation method. The adjoint representation of $SU(3)$ is decomposed as

$$\mathbf{8} \to \mathbf{3} + \mathbf{1} + \mathbf{2} + \mathbf{2} \tag{3.7}$$

in the base of $SU(2)_{13}$. Thus, charges of the generator are given by

$$(\underbrace{+1, -1, 0}_{\mathbf{3}}, \underbrace{0}_{\mathbf{1}}, \underbrace{+1/2, -1/2}_{\mathbf{2}}, \underbrace{+1/2, -1/2}_{\mathbf{2}}), \tag{3.8}$$

since $ad(A_5)$ corresponds to the $U(1)$ charge defined by the τ_1 direction of the $SU(2)_{13}$. The eigenvalues of $D_y(A_5)^2$ for an adjoint field B become

$$2 \times \frac{n^2}{R^2}, \quad \frac{(n \pm a)^2}{R^2}, \quad 2 \times \frac{(n \pm a/2)^2}{R^2}, \tag{3.9}$$

when the eigenfunctions are expanded as $B \propto \cos\frac{ny}{R}, \sin\frac{ny}{R}$. This Kaluza-Klein (KK)[18] expansion applies to the gauge sector (gauge and ghost) and bulk fields sector with $\eta\eta' = +$ in Eqs.(3.4) and (3.5), since their parity eigenvalues, (P, P'), are either $(+,+)$ or $(-,-)$ in this model. Then, the effective potential from an adjoint representational field is given by

$$\begin{aligned} V_{\text{eff}}^{adj(+)} &= \frac{i}{2} \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{1}{2\pi R} \sum_{n=-\infty}^{\infty} \left[\ln\left(-p^2 + \left(\frac{n}{R}\right)^2\right) + \ln\left(-p^2 + \left(\frac{n-a}{R}\right)^2\right) \right. \\ &\qquad \left. + 2\ln\left(-p^2 + \left(\frac{n-a/2}{R}\right)^2\right)\right] \\ &= \frac{C}{2} \sum_{n=1}^{\infty} \frac{1}{n^5} \left[\cos(2\pi n a) + 2\cos(\pi n a)\right], \end{aligned} \tag{3.10}$$

where $C \equiv 3/(64\pi^7 R^5)$. The 2nd equation is derived by the Wick rotation and omitting independent terms of the VEV, a[2, 7]. Equation (3.10) shows the effective potential

†In SUSY case, there appear two doublets as the Wilson line d.o.f..

from one (fermionic) d.o.f. of the field, so that the true effective potential is obtained by producting coefficients as, [(fermionic d.o.f.) − (bosonic d.o.f.)] $\times$ Eq.(3.10). For examples, the gauge sector contribution is $-3\times$ Eq.(3.10), which correctly reproduces the result in Refs.[7, 13]. As for the contributions from bulk fields sector, a Dirac fermion and a complex scalar with $\eta\eta' = +$, are obtained by producting coefficients, 4 and -2, in Eq.(3.10), respectively.

On the other hand, when the adjoint representational bulk fields have $\eta\eta' = -$ in Eqs.(3.4) and (3.5), the eigenfunctions are expanded as $B \propto \cos\frac{(n+1/2)y}{R}, \sin\frac{(n+1/2)y}{R}$ because their parity eigenvalues are $(P,P') = (+,-)$ or $(-,+)$. These half KK expansions induce a sift of VEV $\langle A_5\rangle$ as

$$\left(\frac{\left(n+\frac{1}{2}\right)+Qa}{R}\right) = \left(\frac{n+\left(Qa+\frac{1}{2}\right)}{R}\right), \tag{3.11}$$

where Q is the $U(1)$ charge of definite representation of B. Then, in the case of the adjoint representation with $\eta\eta' = -$, the eigenvalues of $D_y(A_5)^2$ become

$$2\times\frac{(n+1/2)^2}{R^2}, \quad \frac{(n\pm a+1/2)^2}{R^2}, \quad 2\times\frac{(n\pm a/2+1/2)^2}{R^2}, \tag{3.12}$$

which induce

$$\begin{aligned} V_{\rm eff}^{adj(-)} &= \frac{i}{2}\int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{1}{2\pi R}\sum_{n=-\infty}^{\infty}\left[\ln\left(-p^2+\left(\frac{n+1/2}{R}\right)^2\right)\right. \\ &\qquad \left. +\ln\left(-p^2+\left(\frac{n-a+1/2}{R}\right)^2\right) +2\ln\left(-p^2+\left(\frac{n-a/2+1/2}{R}\right)^2\right)\right] \\ &= \frac{C}{2}\sum_{n=1}^{\infty}\frac{1}{n^5}\left[\cos(2\pi n(a-1))+2\cos(\pi n(a-1))\right]. \end{aligned} \tag{3.13}$$

For other representations, the eigenvalues of the covariant derivative can be also evaluated from charges of the same $U(1)$. Let us consider the contribution from a fundamental representational field, for an example. The fundamental representation is decomposed as

$$\mathbf{3} \to \mathbf{2}+\mathbf{1} \tag{3.14}$$

in the base of $SU(2)_{13}$. Then, the $U(1)$ charges corresponding to Eq.(3.8) are

$$(\underbrace{+1/2, -1/2,}_{\mathbf{2}} \underbrace{\ 0\ }_{\mathbf{1}}). \tag{3.15}$$

This means that a fundamental field B with $\eta\eta' = +$ has eigenvalues of $D_y(A_5)^2$ as

$$\frac{(n\pm a)^2}{R^2}, \quad \frac{n^2}{R^2}\ . \tag{3.16}$$

Here eigenfunctions are expanded as $B \propto \cos\frac{ny}{R}, \sin\frac{ny}{R}$, since $(P,P') = (+,+)$ or $(-,-)$. Thus, the effective potential from the fundamental representational field is given by

$$V_{\rm eff}^{fnd(+)} = \frac{C}{2}\sum_{n=1}^{\infty}\frac{1}{n^5}\ \cos(\pi n a). \tag{3.17}$$

In case of $\eta\eta' = -$, eigenfunctions are expanded as $B \propto \cos\frac{(n+1/2)y}{R}, \sin\frac{(n+1/2)y}{R}$, since $(P, P') = (+,-)$ or $(-,+)$. This suggests the eigenvalues of $D_y(A_5)^2$ become

$$\frac{(n \pm a + 1/2)^2}{R}, \quad \frac{(n+1/2)^2}{R}, \tag{3.18}$$

which induces the effective potential,

$$V_{\text{eff}}^{fnd(-)} = \frac{C}{2}\sum_{n=1}^{\infty}\frac{1}{n^5}\ \cos(\pi n(a-1)). \tag{3.19}$$

Now let us show the case when there are $N_a^{(\pm)}$ ($N_f^{(\pm)}$) numbers of adjoint (fundamental) Dirac fermions and $N_s^{(\pm)}$ numbers of complex scalars of fundamental representation in the bulk. The index $(\pm)$ denotes the eigenvalue of $\eta\eta'$. The discussion below Eq.(3.10) suggests the gauge sector induces the effective potential, $-3\times$ Eq.(3.10). While bulk fields contributions are $4N_a^{(+)}\times$ Eq.(3.10), $4N_a^{(-)}\times$ Eq.(3.13), $(4N_f^{(+)} - 2N_s^{(+)})\times$ Eq.(3.17), and $(4N_f^{(-)} - 2N_s^{(-)})\times$ Eq.(3.19). It is because Eqs.(3.10), (3.13), (3.17), and (3.19) show the effective potential from one (fermionic) d.o.f. of the field, so that the true effective potential is obtained by producting coefficients, [(fermionic d.o.f.) $-$ (bosonic d.o.f.)]. Thus, the total effective potential becomes

$$\begin{aligned} V_{\text{eff}} = C\sum_{n=1}^{\infty}\frac{1}{n^5}[(-&\frac{3}{2} + 2N_a^{(+)})\cos(2\pi n a) + 2N_a^{(-)}\cos(\pi n(2a-1)) \\ &+ (-3 + 4N_a^{(+)} - N_s^{(+)} + 2N_f^{(+)})\cos(\pi n a) \\ &+ (4N_a^{(-)} - N_s^{(-)} + 2N_f^{(-)})\cos(\pi n(a-1))], \end{aligned} \tag{3.20}$$

which reproduces the results in Ref.[13]. This counting rule for the coefficients of the effective potential is applicable in general. Thus, we show the effective potential induced from one d.o.f. of each representational field in the following discussions.

The SUSY version of the effective potential can be obtained straightforwardly, when SUSY breaking is induced by the SS mechanism[‡]. For the gauge and ghost contributions, the coefficient should be modified as $-3\times$ Eq.(3.10) $\to$ $-4\times$ Eq.(3.10), and the factor $(1-\cos(2\pi n\beta))$ should be added in the r.h.s. summation in Eq.(3.10). They are coming from *massive* gaugino contributions. The β parameterizes SS SUSY breaking, and 4D effective theory has the gaugino mass of order β/R[2]. As for the bulk fields, they are corresponding to the hypermultiplets of 4D $\mathcal{N}=2$ SUSY. Since one hypermultiplet has one Dirac fermion and two complex scalar d.o.f., and scalar components always have SUSY breaking masses, the SUSY effective potential is obtained by adding the factor $(1-\cos(2\pi n\beta))$ in the summation n of the non-SUSY effective potential induced from the Dirac fermion contributions. By using this technique, we can obtain the effective potential in the SUSY version of this $SU(3)$ model. Considering the situation that $N_f^{(\pm)}$

[‡]In the case of other SUSY breaking, such as introducing explicit soft breaking masses[19], the calculation of the effective potential might be easily done in a similar manner.

and $N_a^{(\pm)}$ species of hypermultiplets of fundamental and adjoint representations in the bulk, respectively, the effective potential becomes

$$
\begin{aligned}
V_{\rm eff} = 2C \sum_{n=1}^{\infty} \frac{1}{n^5}(1-\cos(2\pi n\beta)) \times [(N_a^{(+)}-1)\cos(2\pi n a) + N_a^{(-)}\cos(\pi n(2a-1)) \\
+(2N_a^{(+)}+N_f^{(+)}-2)\cos(\pi n a) + (2N_a^{(-)}+N_f^{(-)})\cos(\pi n(a-1))],
\end{aligned}
\tag{3.21}
$$

which reproduces the result in Ref.[13].

4. More general examples

In this section we show more complicated examples. We calculate the effective potential when there are non-vanishing two or three VEVs in $\langle A_5 \rangle$, and also the case of $P \neq P'$.

4.1 Two VEVs with $P = P'$

Here we show an example of existing two VEVs with $P = P'$ in the $SU(5)$ GUT model[2]. This model has parities,

$$
P = P' = \mathrm{diag}(1,1,1,-1,-1), \tag{4.1}
$$

under which 4D gauge field transforms as

$$
(P,P')(A_\mu) = \left(\begin{array}{ccc|cc} (+,+) & (+,+) & (+,+) & (-,-) & (-,-) \\ (+,+) & (+,+) & (+,+) & (-,-) & (-,-) \\ (+,+) & (+,+) & (+,+) & (-,-) & (-,-) \\ \hline (-,-) & (-,-) & (-,-) & (+,+) & (+,+) \\ (-,-) & (-,-) & (-,-) & (+,+) & (+,+) \end{array} \right). \tag{4.2}
$$

This means that the gauge symmetry is reduced as $SU(5) \to SU(3)_c \times SU(2)_L \times U(1)_Y$. Since the signs of parities in each component of A_5 are completely opposite to those of A_μ as shown in Eqs.(3.2) and (3.3), the zero modes exist in upper-right 3×2 and lower-left 2×3 submatrices corresponding to $(-,-)$ in Eq.(4.2). By using the residual global symmetry, the d.o.f. of Wilson line phases can be set two as

$$
\langle A_5 \rangle = \frac{1}{2gR} \begin{pmatrix} 0 & 0 & 0 & 0 & a \\ 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.3}
$$

In order to obtain the effective potential, it is important to know the eigenvalues of $D_y(A_5)^2$ for the following two generators:

$$
\frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.
$$

They are generators (τ_1) of $SU(2)_{15}$ and $SU(2)_{24}$, respectively. As in the section 3, let us see the decomposition of $SU(5)$ into $SU(2)_{15} \times SU(2)_{24}$.

The adjoint representation of $SU(5)$ is decomposed as

$$\mathbf{24} \to (\mathbf{3},\mathbf{1}) + (\mathbf{1},\mathbf{3}) + 2\times(\mathbf{1},\mathbf{1}) + 2\times(\mathbf{2},\mathbf{1}) + 2\times(\mathbf{1},\mathbf{2}) + 2\times(\mathbf{2},\mathbf{2}). \tag{4.4}$$

This means the eigenvalues of $D_y(A_5)^2$ for the adjoint field are

$$4\times\frac{n^2}{R^2}, \quad \frac{(n\pm a)^2}{R^2}, \quad \frac{(n\pm b)^2}{R^2}, \quad 2\times\frac{(n\pm a/2)^2}{R^2}, \quad 2\times\frac{(n\pm b/2)^2}{R^2}, \quad 2\times\frac{(n\pm(a\pm b)/2)^2}{R^2}, \tag{4.5}$$

for $\cos\frac{ny}{R}$ and $\sin\frac{ny}{R}$ modes. They can be also obtained by the $U(1)$ charges, which are calculated by the commutation relations in the gauge where $\langle A_5\rangle$ is diagonal, $\langle A_5\rangle \propto \mathrm{diag}(a,b,0,-b,-a)$. In this gauge, charge of each component is given by

$$Q(A_\mu) = \begin{pmatrix} 0 & (a-b)/2 & a/2 & (a+b)/2 & a \\ (-a+b)/2 & 0 & b/2 & b & (a+b)/2 \\ -a/2 & -b/2 & 0 & b/2 & a/2 \\ (-a-b)/2 & -b & -b/2 & 0 & (a-b)/2 \\ -a & (-a-b)/2 & -a/2 & (-a+b)/2 & 0 \end{pmatrix}.$$

These eigenvalues suggest that the contribution from one d.o.f. of the adjoint representational field becomes

$$\begin{aligned} V_{\text{eff}}^{adj(+)} &= \frac{C}{2}\sum_{n=1}^{\infty}\frac{1}{n^5}\left[\cos(2\pi na) + \cos(2\pi nb) + 2\cos(\pi na) + 2\cos(\pi nb)\right. \\ &\qquad \left. +2\cos(\pi n(a+b)) + 2\cos(\pi n(a-b))\right]. \end{aligned} \tag{4.6}$$

As for $\overline{\mathbf{5}}$ and $\mathbf{10}$ representations with $\eta\eta' = +$, the decompositions in terms of $SU(2)_{15}\times SU(2)_{24}$ are given as

$$\overline{\mathbf{5}} \to (\mathbf{1},\mathbf{1}) + (\mathbf{2},\mathbf{1}) + (\mathbf{1},\mathbf{2}), \tag{4.7}$$

and

$$\mathbf{10} \to 2\times(\mathbf{1},\mathbf{1}) + (\mathbf{2},\mathbf{1}) + (\mathbf{1},\mathbf{2}) + (\mathbf{2},\mathbf{2}). \tag{4.8}$$

They mean that eigenvalues of $D_y(A_5)^2$ are

$$\frac{n^2}{R^2}, \quad \frac{(n\pm a/2)^2}{R^2}, \quad \frac{(n\pm b/2)^2}{R^2}, \tag{4.9}$$

and

$$2\times\frac{n^2}{R^2}, \qquad \frac{(n\pm a/2)^2}{R^2}, \quad \frac{(n\pm b/2)^2}{R^2}, \quad \frac{(n\pm(a\pm b)/2)^2}{R^2}, \tag{4.10}$$

respectively. Thus, the contributions from one d.o.f. of $\overline{\mathbf{5}}$ and $\mathbf{10}$ with $\eta\eta' = +$ are given by

$$V_{\rm eff}^{\overline{\mathbf{5}}(+)} = \frac{C}{2}\sum_{n=1}^{\infty}\frac{1}{n^5}\left[\cos(\pi n a) + \cos(\pi n b)\right], \tag{4.11}$$

$$V_{\rm eff}^{\mathbf{10}(+)} = \frac{C}{2}\sum_{n=1}^{\infty}\frac{1}{n^5}\left[\cos(\pi n a) + \cos(\pi n b) + \cos(\pi n(a+b)) + \cos(\pi n(a-b))\right], \tag{4.12}$$

respectively, which also reproduce the result in Ref.[2] correctly.

Notice that the contribution from one d.o.f. with $\eta\eta' = -$ is easily obtained by $Qa + Qb \to Qa + Qb + \frac{1}{2}$ in Eqs.(4.6), (4.11), and (4.12) as discussed in section 3. The results are given as

$$\begin{aligned} V_{\rm eff}^{adj(-)} &= \frac{C}{2}\sum_{n=1}^{\infty}\frac{1}{n^5}\left[\cos(\pi n(2a-1)) + \cos(\pi n(2b-1))\right. \\ &\qquad +2\cos(\pi n(a-1)) + 2\cos(\pi n(b-1)) \\ &\qquad \left.+2\cos(\pi n(a+b-1)) + 2\cos(\pi n(a-b-1))\right], \end{aligned} \tag{4.13}$$

$$V_{\rm eff}^{\overline{\mathbf{5}}(-)} = \frac{C}{2}\sum_{n=1}^{\infty}\frac{1}{n^5}\left[\cos(\pi n(a-1)) + \cos(\pi n(b-1))\right], \tag{4.14}$$

$$\begin{aligned} V_{\rm eff}^{\mathbf{10}(-)} &= \frac{C}{2}\sum_{n=1}^{\infty}\frac{1}{n^5}\left[\cos(\pi n(a-1)) + \cos(\pi n(b-1))\right. \\ &\qquad \left.+\cos(\pi n(a+b-1)) + \cos(\pi n(a-b-1))\right]. \end{aligned} \tag{4.15}$$

4.2 Three VEVs with $P = P'$

Next, we show the example of existing three VEVs with $P = P'$ in the $SU(6)$ GUT model. This model has the parities,

$$P = P' = \mathrm{diag}(1,1,1,-1,-1,-1), \tag{4.16}$$

under which 4D gauge field transforms

$$(P,P')(A_\mu) = \left(\begin{array}{ccc|ccc} (+,+) & (+,+) & (+,+) & (-,-) & (-,-) & (-,-) \\ (+,+) & (+,+) & (+,+) & (-,-) & (-,-) & (-,-) \\ (+,+) & (+,+) & (+,+) & (-,-) & (-,-) & (-,-) \\ \hline (-,-) & (-,-) & (-,-) & (+,+) & (+,+) & (+,+) \\ (-,-) & (-,-) & (-,-) & (+,+) & (+,+) & (+,+) \\ (-,-) & (-,-) & (-,-) & (+,+) & (+,+) & (+,+) \end{array}\right). \tag{4.17}$$

This means that the gauge symmetry is reduced as $SU(6) \to SU(3)_c \times SU(3)_L \times U(1)$. Since the signs of parities in each component of A_5 are completely opposite to those of A_μ, the zero modes exist in upper-right 3×3 and lower-left 3×3 submatrices corresponding to $(-,-)$ in Eq.(4.17). By using the residual global symmetry, the d.o.f. of the Wilson line

phases can be set three as

$$\langle A_5 \rangle = \frac{1}{2gR} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & c & 0 & 0 \\ 0 & 0 & c & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 & 0 \end{pmatrix} . \tag{4.18}$$

The effective potential can be calculated in a similar manner as in the section 4.1. The adjoint representation of $SU(6)$ is decomposed as

$$\begin{aligned} \mathbf{35} \to & (\mathbf{3},\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{3},\mathbf{1}) + (\mathbf{1},\mathbf{1},\mathbf{3}) + 2\times(\mathbf{1},\mathbf{1},\mathbf{1}) \\ & +2\times(\mathbf{2},\mathbf{2},\mathbf{1}) + 2\times(\mathbf{1},\mathbf{2},\mathbf{2}) + 2\times(\mathbf{2},\mathbf{1},\mathbf{2}), \end{aligned} \tag{4.19}$$

in terms of $SU(2)_{16} \times SU(2)_{25} \times SU(2)_{34}$. This leads the contribution from one d.o.f. of the adjoint representational field as

$$\begin{aligned} V_{\text{eff}}^{adj(+)} = \frac{C}{2}\sum_{n=1}^{\infty}\frac{1}{n^5} & [\cos(2\pi na) + \cos(2\pi nb) + \cos(2\pi nc) \\ & +2\cos(\pi n(a+b)) + 2\cos(\pi n(a-b)) \\ & +2\cos(\pi n(b+c)) + 2\cos(\pi n(b-c)) \\ & +2\cos(\pi n(c+a)) + 2\cos(\pi n(c-a))] . \end{aligned} \tag{4.20}$$

As for a fundamental representational field with $\eta\eta' = +$, the decomposition is given as following:

$$\mathbf{6} \to (\mathbf{2},\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{2},\mathbf{1}) + (\mathbf{1},\mathbf{1},\mathbf{2})$$

This means the contribution from one d.o.f of the fundamental representational field with $\eta\eta' = +$ is

$$V_{\text{eff}}^{fnd(+)} = \frac{C}{2}\sum_{n=1}^{\infty}\frac{1}{n^5}[\cos(\pi na) + \cos(\pi nb) + \cos(\pi nc)] . \tag{4.21}$$

As for the contribution from fields with $\eta\eta' = -$, the effective potential is obtained by $Qa + Qb + Qc \to Qa + Qb + Qc + \frac{1}{2}$ as discussed in section 3. They are shown as

$$\begin{aligned} V_{\text{eff}}^{adj(-)} = \frac{C}{2}\sum_{n=1}^{\infty}\frac{1}{n^5} & [\cos(\pi n(2a-1)) + \cos(\pi n(2b-1)) + \cos(\pi n(2c-1)) \\ & +2\cos(\pi n(a+b-1)) + 2\cos(\pi n(a-b-1)) \\ & +2\cos(\pi n(b+c-1)) + 2\cos(\pi n(b-c-1)) \\ & +2\cos(\pi n(c+a-1)) + 2\cos(\pi n(c-a-1))] , \end{aligned} \tag{4.22}$$

$$V_{\text{eff}}^{fnd(-)} = \frac{C}{2}\sum_{n=1}^{\infty}\frac{1}{n^5}[\cos(\pi n(a-1)) + \cos(\pi n(b-1)) + \cos(\pi n(c-1))] . \tag{4.23}$$

In section 4.1 and 4.2, we have shown the $P = P'$ case, where only either $(+,+)$ and $(-,-)$ modes (in gauge sector and bulk fields sector with $\eta\eta' = +$), or $(+,-)$ and

$(-,+)$ modes (in bulk fields sector with $\eta\eta' = -$) exist in each representation. In section 4.3, we will show an example of $P \neq P'$ case, where all modes of $(\pm,\pm)$ can exist in one representation.

4.3 One VEV with $P \neq P'$

Now let us show an example of existing one VEV with $P \neq P'$ in the $SU(6)$ GUT model[8, 13]. This model has the parities,

$$\begin{aligned} P &= \mathrm{diag}(1,1,1,1,-1,-1) \\ P' &= \mathrm{diag}(1,-1,-1,-1,-1,-1), \end{aligned} \tag{4.24}$$

under which 4D gauge field transforms

$$(P,P')(A_\mu) = \left(\begin{array}{c|ccc|cc} (+,+) & (+,-) & (+,-) & (+,-) & (-,-) & (-,-) \\ \hline (+,-) & (+,+) & (+,+) & (+,+) & (-,+) & (-,+) \\ (+,-) & (+,+) & (+,+) & (+,+) & (-,+) & (-,+) \\ (+,-) & (+,+) & (+,+) & (+,+) & (-,+) & (-,+) \\ \hline (-,-) & (-,+) & (-,+) & (-,+) & (+,+) & (+,+) \\ (-,-) & (-,+) & (-,+) & (-,+) & (+,+) & (+,+) \end{array}\right). \tag{4.25}$$

This means that the gauge symmetry is reduced as $SU(6) \to SU(3)_c \times SU(2)_L \times U(1)_Y \times U(1)$. Since the signs of parities in each component of A_5 are completely opposite to those of A_μ, the zero modes exist in upper-right 1×2 and lower-left 2×1 submatrices corresponding to $(-,-)$ in Eq.(4.25). This zero mode is regarded as a "Higgs doublet" in the gauge-Higgs unified models[8, 13]. By using the residual global symmetry, the d.o.f. of the Wilson line phase can be set just one as

$$\langle A_5 \rangle = \frac{1}{2gR} \begin{pmatrix} 0&0&0&0&0&a \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ a&0&0&0&0&0 \end{pmatrix}. \tag{4.26}$$

We can always take the gauge, in which this VEV becomes diagonal as $\langle A_5 \rangle \propto \mathrm{diag}(1,0,0,0,0,-1)$. The $U(1)$ charge, Q, for this direction and the eigenvalue of PP' are given by

$$(Q,PP')(A_\mu) = \begin{pmatrix} (0,+) & (\frac{1}{2},-) & (\frac{1}{2},-) & (\frac{1}{2},-) & (\frac{1}{2},+) & (1,+) \\ (-\frac{1}{2},-) & (0,+) & (0,+) & (0,+) & (0,-) & (\frac{1}{2},-) \\ (-\frac{1}{2},-) & (0,+) & (0,+) & (0,+) & (0,-) & (\frac{1}{2},-) \\ (-\frac{1}{2},-) & (0,+) & (0,+) & (0,+) & (0,-) & (\frac{1}{2},-) \\ (-\frac{1}{2},+) & (0,-) & (0,-) & (0,-) & (0,+) & (\frac{1}{2},+) \\ (-1,+) & (-\frac{1}{2},-) & (-\frac{1}{2},-) & (-\frac{1}{2},-) & (-\frac{1}{2},+) & (0,+) \end{pmatrix}. \tag{4.27}$$

This means that the eigenvalues of $D_y(A_5)^2$ are

$$11 \times \frac{n^2}{R^2},\ \ 6 \times \frac{(n+1/2)^2}{R^2},\ \ \frac{(n \pm a)^2}{R},\ \ 2 \times \frac{(n \pm a/2)^2}{R^2},\ \ 6 \times \frac{(n \pm a/2 + 1/2)^2}{R^2}, \tag{4.28}$$

since the eigenfunctions are expanded as $B \propto \cos\frac{ny}{R}, \sin\frac{ny}{R}$ $(\cos\frac{(n+1/2)y}{R}, \sin\frac{(n+1/2)y}{R})$ for $PP' = +$ $(PP' = -)$. This case is applicable for the gauge sector and bulk fields sector with $\eta\eta' = +$. Equation (4.28) suggests that the effective potential for one d.o.f. is given by

$$V_{\rm eff}^{adj(+)} = \frac{C}{2}\sum_{n=1}^{\infty}\frac{1}{n^5}\left[6\cos(n\pi(a-1)) + 2\cos(n\pi a) + \cos(2n\pi a)\right]. \tag{4.29}$$

We can also reach this conclusion via an analysis independent of matrix representation. The product of parities (Eq.(4.24)),

$$PP' = {\rm diag}(1, -1, -1, -1, 1, 1), \tag{4.30}$$

by which the gauge symmetry is reduced as $SU(6) \to SU(3)_c \times SU(3)_L \times U(1)$, plays an important role. The adjoint representation is decomposed as

$$\mathbf{35} \to (\mathbf{8}, \mathbf{1}, +) + (\mathbf{1}, \mathbf{8}, +) + (\mathbf{1}, \mathbf{1}, +) + (\mathbf{3}, \overline{\mathbf{3}}, -) + (\overline{\mathbf{3}}, \mathbf{3}, -) \tag{4.31}$$

in terms of $(SU(3)_c,\ SU(3)_L,\ PP')$. Since $PP' = +$ for the Wilson line phases, these d.o.f. are proportional to some of the generators of the reduced symmetry. In particular, in this case, the VEV is proportional to one generator of $SU(3)_L$. Then, a similar analysis as in section 3 leads the above effective potential.

As for the case of $\eta\eta' = -$, the eigenvalues of $D_y(A_5)^2$ in bulk fields sector become

$$11\times\frac{(n+1/2)^2}{R^2},\ \ 6\times\frac{n^2}{R^2},\ \ \frac{(n \pm a + 1/2)^2}{R},\ \ 2\times\frac{(n \pm a/2 + 1/2)^2}{R^2},\ \ 6\times\frac{(n \pm a/2)^2}{R^2}, \tag{4.32}$$

which lead the effective potential,

$$V_{\rm eff}^{adj(-)} = \frac{C}{2}\sum_{n=1}^{\infty}\frac{1}{n^5}\left[6\cos(n\pi a) + 2\cos(n\pi(a-1)) + \cos(n\pi(2a-1))\right]. \tag{4.33}$$

For a fundamental representational field with $\eta\eta' = +$, the $U(1)$ charge and PP' are given by

$$((1/2, +), (0, -), (0, -), (0, -), (0, +), (-1/2, +))^T.$$

This means that the eigenvalues are

$$\frac{n^2}{R^2},\ \ 3 \times \frac{(n+1/2)^2}{R^2},\ \ \frac{(n \pm a/2)^2}{R^2}, \tag{4.34}$$

which derive the effective potential,

$$V_{\rm eff}^{fnd(+)} = \frac{C}{2}\sum_{n=1}^{\infty}\frac{1}{n^5}\left[\cos(n\pi a))\right]. \tag{4.35}$$

On the other hand, a fundamental representational field with $\eta\eta' = -$ has

$$((1/2,-),(0,+),(0,+),(0,+),(0,-),(-1/2,-))^T,$$

which means the eigenvalues become

$$\frac{(n+1/2)^2}{R^2}, \quad 3\times\frac{n^2}{R^2}, \quad \frac{(n\pm a/2+1/2)^2}{R^2}. \tag{4.36}$$

Then the effective potential is given by

$$V_{\rm eff}^{fnd(-)} = \frac{C}{2}\sum_{n=1}^{\infty}\frac{1}{n^5}\left[\cos(n\pi(a-1))\right]. \tag{4.37}$$

Above calculations reproduce the results in Ref.[13].

5. The general formula

Now let us show the general formula of the effective potential in the 5D $SU(N)$ gauge theory with the general boundary conditions. In general, the parity operators, P and P', are shown as

$$\begin{aligned} P &= {\rm diag}(+,\cdots,+,+,\cdots,+,-,\cdots,-,-,\cdots,-), \\ P' &= {\rm diag}(\underbrace{+,\cdots,+}_{n_+^+},\underbrace{-,\cdots,-}_{n_+^-},\underbrace{+,\cdots,+}_{n_-^+},\underbrace{-,\cdots,-}_{n_-^-}), \end{aligned} \tag{5.1}$$

where $N = n_-^+ + n_+^- + n_-^+ + n_-^-$. Under the parities, (P,P'), the gauge field, A_μ, transforms as

$$\begin{array}{c c} & \begin{array}{cccc} n_+^+ & n_+^- & n_-^+ & n_-^- \end{array} \\ \begin{array}{c} n_+^+ \\ n_+^- \\ n_-^+ \\ n_-^- \end{array} & \begin{pmatrix} (+,+) & (+,-) & (-,+) & (-,-) \\ (+,-) & (+,+) & (-,-) & (-,+) \\ (-,+) & (-,-) & (+,+) & (+,-) \\ (-,-) & (-,+) & (+,-) & (+,+) \end{pmatrix} \end{array} \tag{5.2}$$

which means that $SU(N)$ is broken into $SU(n_+^+)\times SU(n_+^-)\times SU(n_-^+)\times SU(n_-^-)\times U(1)^3$. Here, $(+,-)$ parts and $(-,+)$ parts have half KK-mode expansion.

The d.o.f. of the Wilson line phases are reside in $(-,-)$ parts in Eq.(5.2), which are shown as

$$\langle A_5\rangle = \frac{1}{2gR}\begin{pmatrix} 0 & 0 & 0 & \Theta_a \\ 0 & 0 & \Theta_b & 0 \\ 0 & \Theta_b^\dagger & 0 & 0 \\ \Theta_a^\dagger & 0 & 0 & 0 \end{pmatrix}. \tag{5.3}$$

The residual gauge freedom reduces the number of the d.o.f. of the Wilson line phases as

$$\min(n_+^+, n_-^-) + \min(n_+^-, n_-^+). \tag{5.4}$$

Hereafter, we denote $\min(n_+^+, n_-^-)$ and $\min(n_+^-, n_-^+)$ as A and B, respectively. For example, when $n_+^+ < n_-^-$, Θ_a can be transformed into following form.

$$\Theta_a = \begin{pmatrix} a_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_A & 0 & \cdots & 0 \end{pmatrix} \tag{5.5}$$

Similarly, when $n_+^- < n_-^+$, Θ_b can be transformed into following form.

$$\Theta_b = \begin{pmatrix} b_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & b_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_B & 0 & \cdots & 0 \end{pmatrix} \tag{5.6}$$

Let us pick up two non-vanishing VEVs, *e.g.* a_1 and a_2. In this case, it is useful to decompose $SU(N)$ into $SU(N-4) \times SU(2)_1 \times SU(2)_2$, where $SU(2)_i$ is determined by the "position" of VEV, a_i, in the $SU(N)$ base. The adjoint representation of $SU(N)$ is decomposed as

$$\begin{aligned} adj. &= (adj., \mathbf{1}) + (\mathbf{1}, \mathbf{3}) + (\mathbf{1}, \mathbf{1}) + (fnd., \mathbf{2}) + (\overline{fnd.}, \mathbf{2}) \\ &= [(adj., \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{1}) + (fnd., \mathbf{2}, \mathbf{1}) + (\overline{fnd.}, \mathbf{2}, \mathbf{1})] \\ &\quad + [(\mathbf{1}, \mathbf{1}, \mathbf{3})] + [(\mathbf{1}, \mathbf{1}, \mathbf{1})] \\ &\quad + [(fnd., \mathbf{1}, \mathbf{2}) + (\mathbf{1}, \mathbf{2}, \mathbf{2})] + [(\overline{fnd.}, \mathbf{1}, \mathbf{2}) + (\mathbf{1}, \mathbf{2}, \mathbf{2})]. \end{aligned} \tag{5.7}$$

This suggests the non-vanishing eigenvalues of (T_1^1, T_2^1) are

$$(\pm 1, 0),\ (0, \pm 1),\ 2 \times \left(\pm\frac{1}{2}, \pm\frac{1}{2}\right),\ (N-4) \times \left(\pm\frac{1}{2}, 0\right),\ (N-4) \times \left(0, \pm\frac{1}{2}\right), \tag{5.8}$$

where T_i^1 is the 1st generator (τ_1) of $SU(2)_i$. Furthermore, if these two VEVs reside in the same Θ_i, the components with eigenvalues $(\pm\frac{1}{2}, \pm\frac{1}{2})$ exist in the part of $(P, P') = (+,+)$ or $(-,-)$, and therefore have integer KK-expansion. On the other hand, if these two VEVs reside in different Θ_i, such components exist in the part of $(P, P') = (+,-)$ or $(-,+)$, and therefore have half KK-expansion. When we deal with more than two non-vanishing VEVs, above observation dealing with two VEVs case is useful. By taking combinations

of above decompositions, we obtain the following general effective potential,[§]

$$\begin{aligned}
V_{\rm eff}^{adj(+)} &= \frac{C}{2}\sum_{n=1}^{\infty}\frac{1}{n^5}\left[\sum_{i,j}^{A}\cos(n\pi(a_i\pm a_j))+\sum_{i,j}^{B}\cos(n\pi(b_i\pm b_j))\right.\\
&+2\sum_{i}^{A}\sum_{j}^{B}\cos(n\pi(a_i\pm b_j-1))\\
&+2\left|n_+^+-n_-^-\right|\left(\sum_{i}^{A}\cos(n\pi a_i)+\sum_{i}^{B}\cos(n\pi(b_i-1))\right)\\
&\left.+2\left|n_+^--n_-^+\right|\left(\sum_{i}^{A}\cos(n\pi(a_i-1))+\sum_{i}^{B}\cos(n\pi b_i)\right)\right], \qquad (5.9)
\end{aligned}$$

for one d.o.f. of an adjoint representational field. As discussed in section 3, the true effective potential is obtained by producting coefficients as, [(fermionic d.o.f.) $-$ (bosonic d.o.f.)] $\times$ Eq.(5.9). Especially, the gauge and ghost contributions are obtained by $-3\times$ Eq.(5.9).

As for the contribution from an adjoint representational field with $\eta\eta'=-$, it is obtained by modifying $Qa_i+Qb_i\to Qa_i+Qb_i+1/2$, as discussed in section 3. The result is

$$\begin{aligned}
V_{\rm eff}^{adj(-)} &= \frac{C}{2}\sum_{n=1}^{\infty}\frac{1}{n^5}\left[\sum_{i,j}^{A}\cos(n\pi(a_i\pm a_j-1))+\sum_{i,j}^{B}\cos(n\pi(b_i\pm b_j-1))\right.\\
&+2\sum_{i}^{A}\sum_{j}^{B}\cos(n\pi(a_i\pm b_j))\\
&+2\left|n_+^+-n_-^-\right|\left(\sum_{i}^{A}\cos(n\pi(a_i-1))+\sum_{i}^{B}\cos(n\pi b_i)\right)\\
&\left.+2\left|n_+^--n_-^+\right|\left(\sum_{i}^{A}\cos(n\pi a_i)+\sum_{i}^{B}\cos(n\pi(b_i-1))\right)\right]. \qquad (5.10)
\end{aligned}$$

Next, let us calculate the contribution from a fundamental representational field with $\eta\eta'=+$. The parity of the fundamental representation of $SU(N)$, (P,P'), is denoted as

$$\begin{pmatrix}(+,+)\\(+,-)\\(-,+)\\(-,-)\end{pmatrix}\begin{array}{l}n_+^+\\n_+^-\\n_-^+\\n_-^-\end{array}. \qquad (5.11)$$

[§]For $A=0$, $\sum_{i}^{A}$ means zero.

The fundamental representation is decomposed as

$$
\begin{aligned}
fnd. &= (fnd., \mathbf{1}) + (\mathbf{1}, \mathbf{2}), \\
&= [(fnd., \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{2}, \mathbf{1})] + (\mathbf{1}, \mathbf{1}, \mathbf{2}),
\end{aligned}
\tag{5.12}
$$

under the representations of $SU(N-4) \times SU(2)_1 \times SU(2)_2$. Taking account how the VEV $\langle A_5 \rangle$ acts on the fundamental representation, Eq.(5.11), we can calculate the contribution from a fundamental representational field with $\eta\eta' = +$ as

$$
V_{\text{eff}}^{fnd(+)} = \frac{C}{2} \sum_{n=1}^{\infty} \frac{1}{n^5} \left[\sum_{i}^{A} \cos(n\pi a_i) + \sum_{i}^{B} \cos(n\pi(b_i - 1)) \right], \tag{5.13}
$$

and with $\eta\eta' = -$ as

$$
V_{\text{eff}}^{fnd(-)} = \frac{C}{2} \sum_{n=1}^{\infty} \frac{1}{n^5} \left[\sum_{i}^{A} \cos(n\pi(a_i - 1)) + \sum_{i}^{B} \cos(n\pi b_i) \right]. \tag{5.14}
$$

For the (anti-)symmetric tensorial representation of $SU(N)$, they are decomposed as

$$
\begin{aligned}
\begin{smallmatrix}\square\\\square\end{smallmatrix} &= (\begin{smallmatrix}\square\\\square\end{smallmatrix}, \mathbf{1}) + (fnd., \mathbf{2}) + (\mathbf{1}, \mathbf{1}) \\
&= [(\begin{smallmatrix}\square\\\square\end{smallmatrix}, \mathbf{1}, \mathbf{1}) + (fnd., \mathbf{2}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{1})] \\
&\quad + [(fnd., \mathbf{1}, \mathbf{2}) + (\mathbf{1}, \mathbf{2}, \mathbf{2})] + (\mathbf{1}, \mathbf{1}, \mathbf{1}),
\end{aligned}
\tag{5.15}
$$

$$
\begin{aligned}
\square\!\square &= (\square\!\square, \mathbf{1}) + (fnd., \mathbf{2}) + (\mathbf{1}, \mathbf{3}) \\
&= [(\square\!\square, \mathbf{1}, \mathbf{1}) + (fnd., \mathbf{2}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}, \mathbf{1})] \\
&\quad + [(fnd., \mathbf{1}, \mathbf{2}) + (\mathbf{1}, \mathbf{2}, \mathbf{2})] + (\mathbf{1}, \mathbf{1}, \mathbf{3}),
\end{aligned}
\tag{5.16}
$$

in the base of $SU(N-4) \times SU(2)_1 \times SU(2)_2$. The half KK-modes are distributed in a similar way as in the adjoint representation case. The effective potential of the anti-symmetric tensor field with $\eta\eta' = \pm$ becomes[¶]

$$
\begin{aligned}
V_{\text{eff}}^{\begin{smallmatrix}\square\\\square\end{smallmatrix}(+)} &= \frac{C}{2} \sum_{n=1}^{\infty} \frac{1}{n^5} \left[\sum_{i>j}^{A} \cos(n\pi(a_i \pm a_j)) + \sum_{i>j}^{B} \cos(n\pi(b_i \pm b_j)) \right. \\
&\quad + \sum_{i}^{A} \sum_{j}^{B} \cos(n\pi(a_i \pm b_j - 1)) \\
&\quad + \left| n_+^+ - n_-^- \right| \left(\sum_{i}^{A} \cos(n\pi a_i) + \sum_{i}^{B} \cos(n\pi(b_i - 1)) \right) \\
&\quad \left. + \left| n_+^- - n_-^+ \right| \left(\sum_{i}^{A} \cos(n\pi(a_i - 1)) + \sum_{i}^{B} \cos(n\pi b_i) \right) \right],
\end{aligned}
\tag{5.17}
$$

¶For $A = 0, 1$, $\sum_{i>j}^{A}$ represents zero.

and

$$
\begin{aligned}
V_{\rm eff}^{\begin{smallmatrix}\Box\\ \Box\end{smallmatrix}\,(-)} = \frac{C}{2}\sum_{n=1}^{\infty}\frac{1}{n^5}\Bigg[&\sum_{i>j}^{A}\cos(n\pi(a_i \pm a_j - 1)) + \sum_{i>j}^{B}\cos(n\pi(b_i \pm b_j - 1)) \\
&+ \sum_{i}^{A}\sum_{j}^{B}\cos(n\pi(a_i \pm b_j)) \\
&+ \left|n_{+}^{+} - n_{-}^{-}\right|\left(\sum_{i}^{A}\cos(n\pi(a_i - 1)) + \sum_{i}^{B}\cos(n\pi b_i)\right) \\
&+ \left|n_{+}^{-} - n_{-}^{+}\right|\left(\sum_{i}^{A}\cos(n\pi a_i) + \sum_{i}^{B}\cos(n\pi(b_i - 1))\right)\Bigg].
\end{aligned}
\tag{5.18}
$$

On the other hand, the effective potential of the symmetric tensor with $\eta\eta' = \pm$ becomes

$$
\begin{aligned}
V_{\rm eff}^{\Box\!\Box(+)} = \frac{C}{2}\sum_{n=1}^{\infty}\frac{1}{n^5}\Bigg[&\sum_{i}^{A}\cos(2n\pi a_i) + \sum_{i}^{B}\cos(2n\pi b_i) \\
&+ \sum_{i>j}^{A}\cos(n\pi(a_i \pm a_j)) + \sum_{i>j}^{B}\cos(n\pi(b_i \pm b_j)) + \sum_{i}^{A}\sum_{j}^{B}\cos(n\pi(a_i \pm b_j - 1)) \\
&+ \left|n_{+}^{+} - n_{-}^{-}\right|\left(\sum_{i}^{A}\cos(n\pi a_i) + \sum_{i}^{B}\cos(n\pi(b_i - 1))\right) \\
&+ \left|n_{+}^{-} - n_{-}^{+}\right|\left(\sum_{i}^{A}\cos(n\pi(a_i - 1)) + \sum_{i}^{B}\cos(n\pi b_i)\right)\Bigg],
\end{aligned}
\tag{5.19}
$$

and

$$
\begin{aligned}
V_{\rm eff}^{\Box\!\Box(-)} = \frac{C}{2}\sum_{n=1}^{\infty}\frac{1}{n^5}\Bigg[&\sum_{i}^{A}\cos(2n\pi(a_i - \frac{1}{2})) + \sum_{i}^{B}\cos(2n\pi(b_i - \frac{1}{2})) \\
&+ \sum_{i>j}^{A}\cos(n\pi(a_i \pm a_j - 1)) + \sum_{i>j}^{B}\cos(n\pi(b_i \pm b_j - 1)) \\
&+ \sum_{i}^{A}\sum_{j}^{B}\cos(n\pi(a_i \pm b_j)) \\
&+ \left|n_{+}^{+} - n_{-}^{-}\right|\left(\sum_{i}^{A}\cos(n\pi(a_i - 1)) + \sum_{i}^{B}\cos(n\pi b_i)\right) \\
&+ \left|n_{+}^{-} - n_{-}^{+}\right|\left(\sum_{i}^{A}\cos(n\pi a_i) + \sum_{i}^{B}\cos(n\pi(b_i - 1))\right)\Bigg].
\end{aligned}
\tag{5.20}
$$

As for the calculation of SUSY version, the effective potential can be obtained straightforwardly as discussed in section 3. The gauge sector contributions are obtained by adding the coefficient, -4, and the factor, $(1 - \cos(2\pi n\beta))$, in the r.h.s. summation of Eq.(5.9).

Here, the factor -4 denotes the number of the (bosonic) d.o.f. of massless modes and $(1-\cos(2\pi n\beta))$ is due to the *massive* gaugino contributions possessing $\mathcal{O}(\beta/R)$ SUSY breaking masses. As for the bulk fields contributions, the effective potential of the SUSY version is obtained by adding the factor $(1-\cos(2\pi n\beta))$ in the summation n in the non-SUSY effective potential induced from the Dirac fermion of a representation.

For the contributions from higher representations, we can calculate in the same way, since our method only uses the group theoretical analysis.

6. Summary and discussion

We show the general formula of the one loop effective potential of the 5D $SU(N)$ gauge theory compactified on an orbifold, S^1/Z_2. The formula shows the case when there are fundamental, (anti-)symmetric tensor, adjoint representational bulk fields. Our calculation method is also applicable when there are higher representational bulk fields. The SUSY version of the effective potential with SS breaking can be obtained straightforwardly. We have also shown some examples of effective potentials in cases of one VEV with $P=P'$ in $SU(3)$, two VEVs with $P=P'$ in $SU(5)$, three VEVs with $P=P'$ in $SU(6)$ and one VEV with $P\neq P'$ in $SU(6)$. All of which are reproduced by our general formula.

We emphasize our method can be also applied to models with a gauge symmetry other than $SU(N)$, such as $SO(10)$ or E_6. It has been difficult to analyze the vacuum structure in those models, because of the hard task of calculating the complicated commutation relations. However, our calculation method make it possible to analyze the vacuum structure even in those models. We expect novel models can be built through such researches. We will investigate this possibility in another paper.

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